Why Mu is μseful

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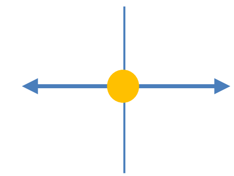
Introduction

- This is a slightly modified presentation I gave to the AFRC controls branch in 2014 to promote Mu (μ) for controller evaluation before flight testing
  - The main goals of this presentation was to discuss how μ analysis
    - Can be understood and how μ is related to Nyquist theory
    - Can be a tool which can justify confidence in typical margin analysis methods
    - Can be physically understandable
- Historically
  - Gain and phase margins are used to measure robustness of a controller because
    - There are industry standards
    - They are generally well understood and have physical significance
      - i.e. Multiply a loop by 2 to show the system goes unstable

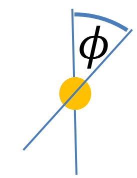
Understanding Gain and Phase Margins

- Gain and phase margins are measures of robustness to real gain **or** phase parameter uncertainty and tell
  - How much gain increase/decrease on a single loop can be tolerated

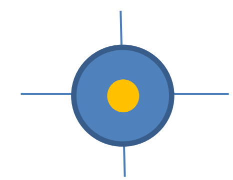


OR

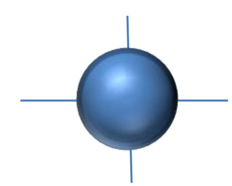
  - How much phase lag/lead on a single loop can be tolerated



- But…
  - Is true environmental uncertainty just gain **or** phase uncertainty?
  - It is typically both simultaneously (complex uncertainty)



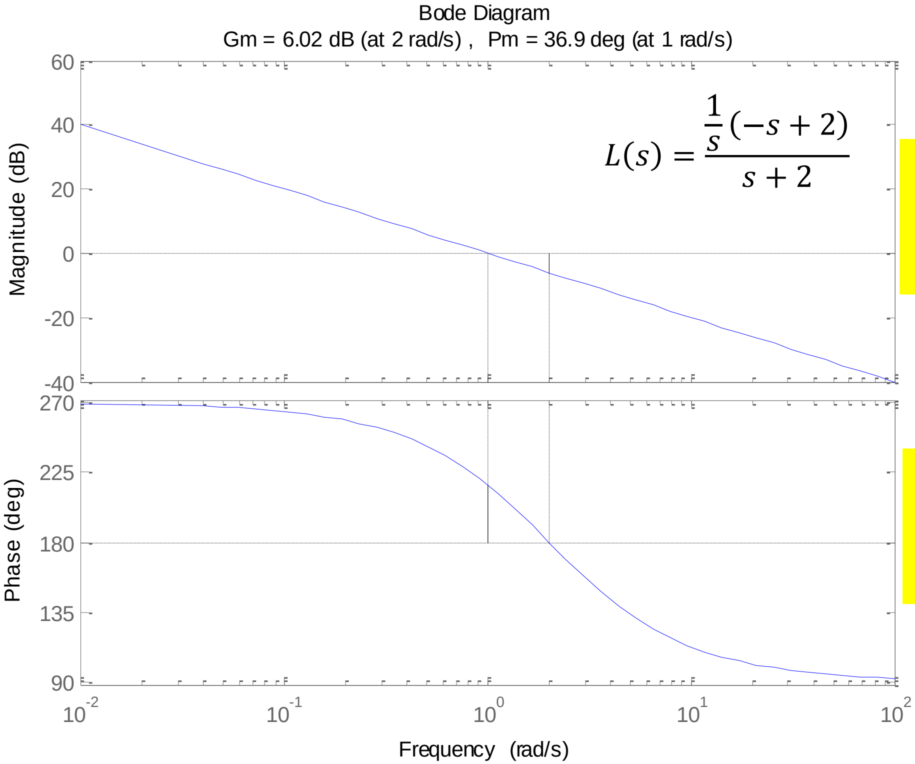
  - SISO margins do not account for un-modeled dynamics or simultaneous loop closure uncertainties



μ analysis supports a broader definition of uncertainty

Gain and phase margin analysis

- Plant: Stable, non-minimum phase
- The gain margin is 6.02 dB (or 2) at 2 rad/s
- The phase margin is 36.9 deg at 1 rad/s



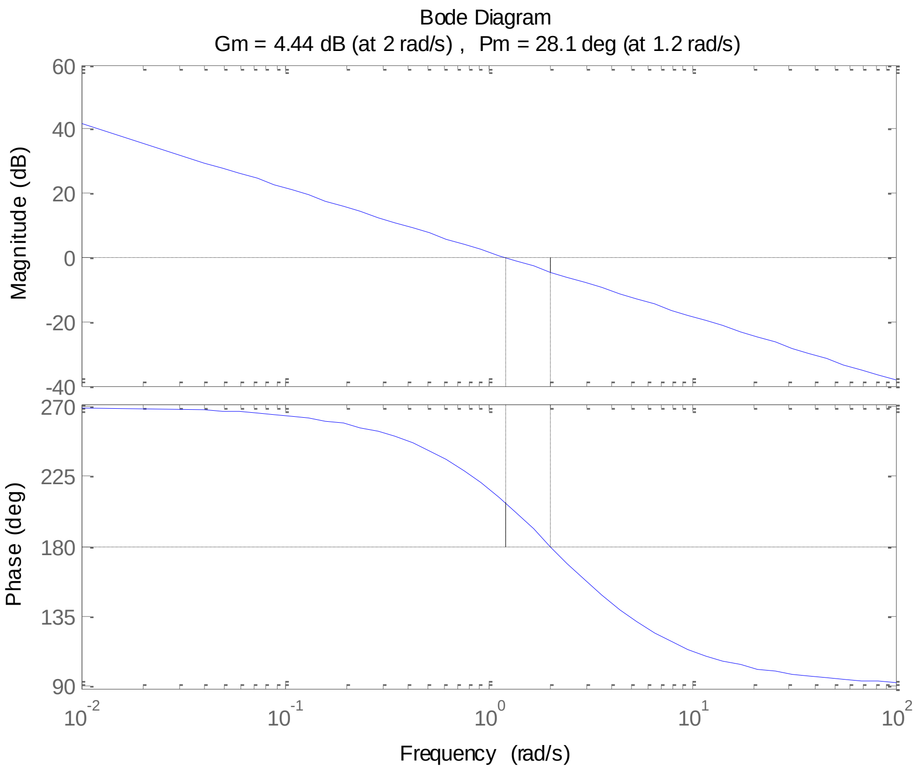
Gain margin is computed with phase fixed

Phase margin is computed with gain fixed

One Gain and Phase Margin Analysis Shortcoming

- What if gain on $L(s)$ is increased by 20%, what is the phase margin?
  - Was 36.9 deg before
  - Now 28.1 deg

$$L(s) = 1.2 * \frac{\frac{1}{s}(-s+2)}{s+2}$$



While physical insight is gained from this analysis, it is unreliable for simultaneous gain **and** phase perturbations.

How μ analysis is Presented

- Generalized Nyquist stability criterion
- Derivation of the required $M\Delta$ structure
- Theoretical computation of μ
- What μ tells us and how it compares to standard margin methods
- Everything is presented without proof but is drawn from Skogestad and Postlethwaite

Sigurd Skogestad and Ian Postlethwaite, Multivariable Feedback Control Analysis and Design

Ref:
Multivariable Feedback Control
Skogestad and Postlethwaite

Nyquist Stability (SISO) Theorem

- Consider a feedback system with asymptotically stable open loop $L(s) = G(s)K(s)$
  - The feedback system is asymptotically stable if $1 + L(s)$ does not encircle the origin
- If $L(s)$ has P_{ol} unstable poles
  - The feedback system is stable iff the Nyquist diagram makes P_{ol} anti-clockwise encirclements of the origin

Note: If $L(s)$ is used, then $L(s)$ must not encircle the point -1. The reason 1 has been added is to show its similarity to the generalized Nyquist theorem

Ref:
Multivariable Feedback Control
Skogestad and Postlethwaite

Generalized (MIMO) Nyquist theorem

- Consider a feedback system with asymptotically stable open loop $L(s)$
  - The feedback system is asymptotically stable if $\det(I + L(s))$ does not encircle the origin
- If $L(s)$ has P_{ol} unstable poles in $L(s)$
  - The closed loop transfer function with $L(s)$ and negative feedback is stable iff Nyquist plot of $\det\big(I + L(s)\big)$
    - Makes P_{ol} anti-clockwise encirclements of the origin
    - Does not pass through the origin

$$\det\big(I + L(s)\big) = 0$$

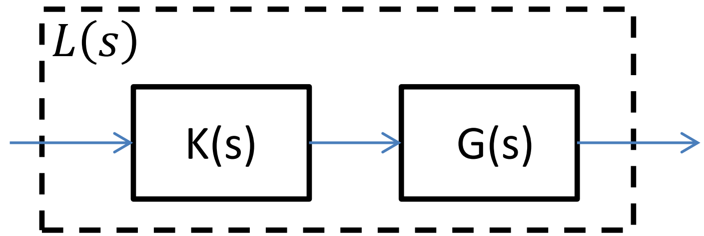
Major difference is the determinant when going from SISO to MIMO

Physical Considerations of the Generalized Nyquist theorem

- For stable or unstable $L(s)$, the determinant of $I + L(s)$ should not be zero
- If $\det\left(I + L(s)\right) = 0$ then it will not have an inverse
  - The problem is not well-posed and there is not a unique solution for input and output signals to $\left(I + L(s)\right)^{-1}$

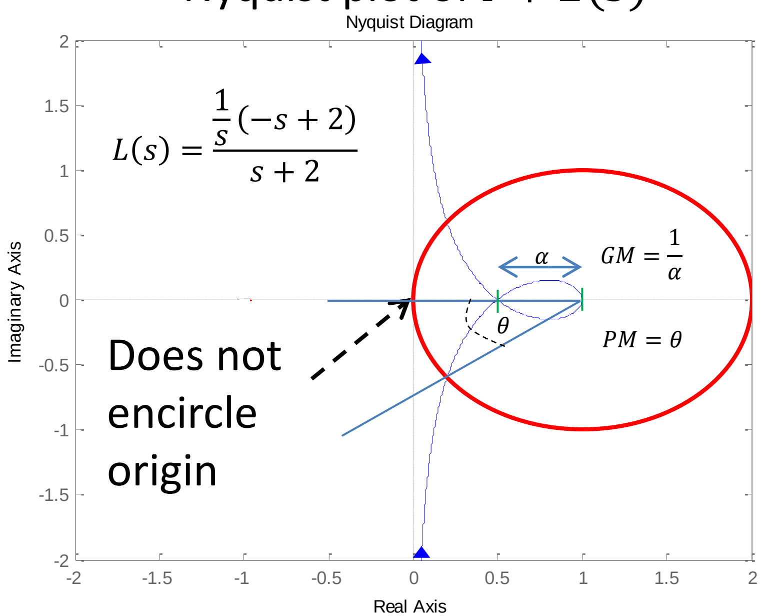
Bottom line: $\det\left(I + L(s)\right)$ must be non-zero over all frequencies. The shortest distance to zero on the complex plane is a margin directly related to Mu (or μ)

Comparison of SISO and MIMO Nyquist stability criteria



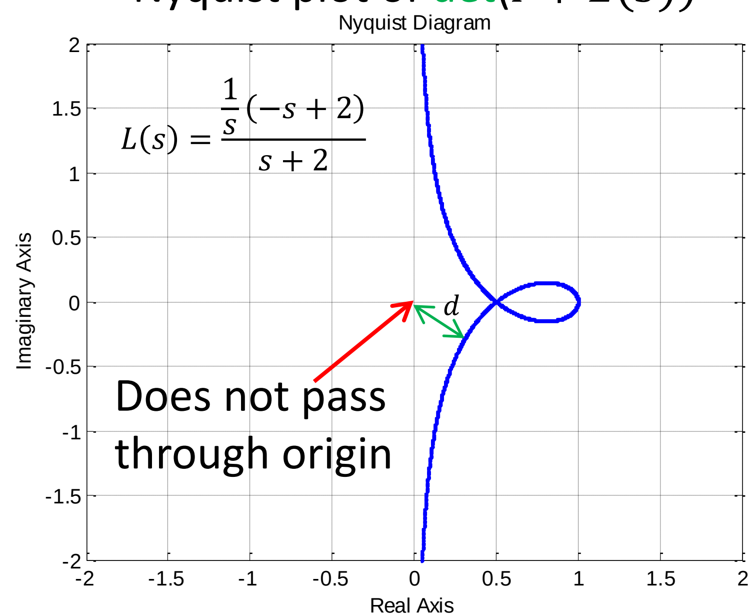
SISO criterion

Nyquist plot of $I + L(s)$



MIMO criterion

Nyquist plot of det$(I + L(s))$

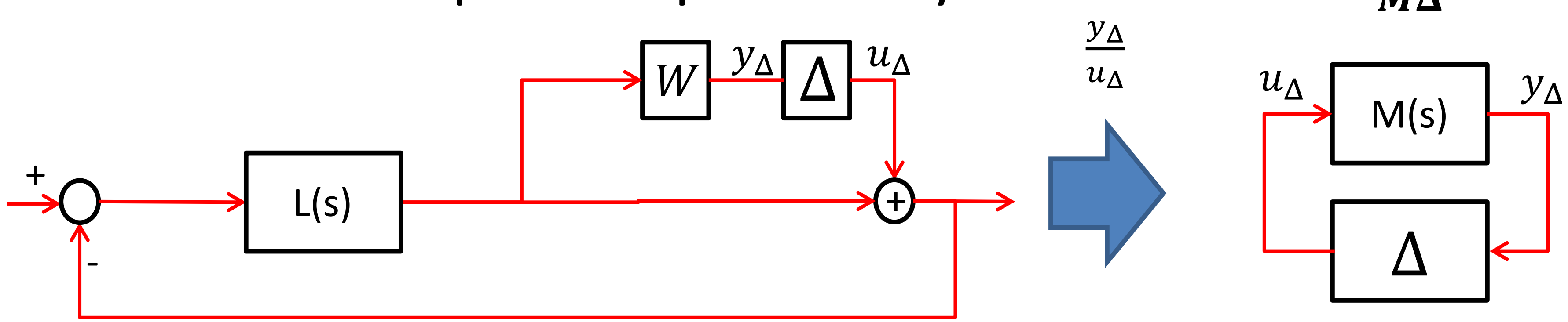


Therefore $\frac{L(s)}{I+L(s)}$ is asymptotically stable (i.e. all poles in the LHP)

How Uncertain Systems can be Analyzed

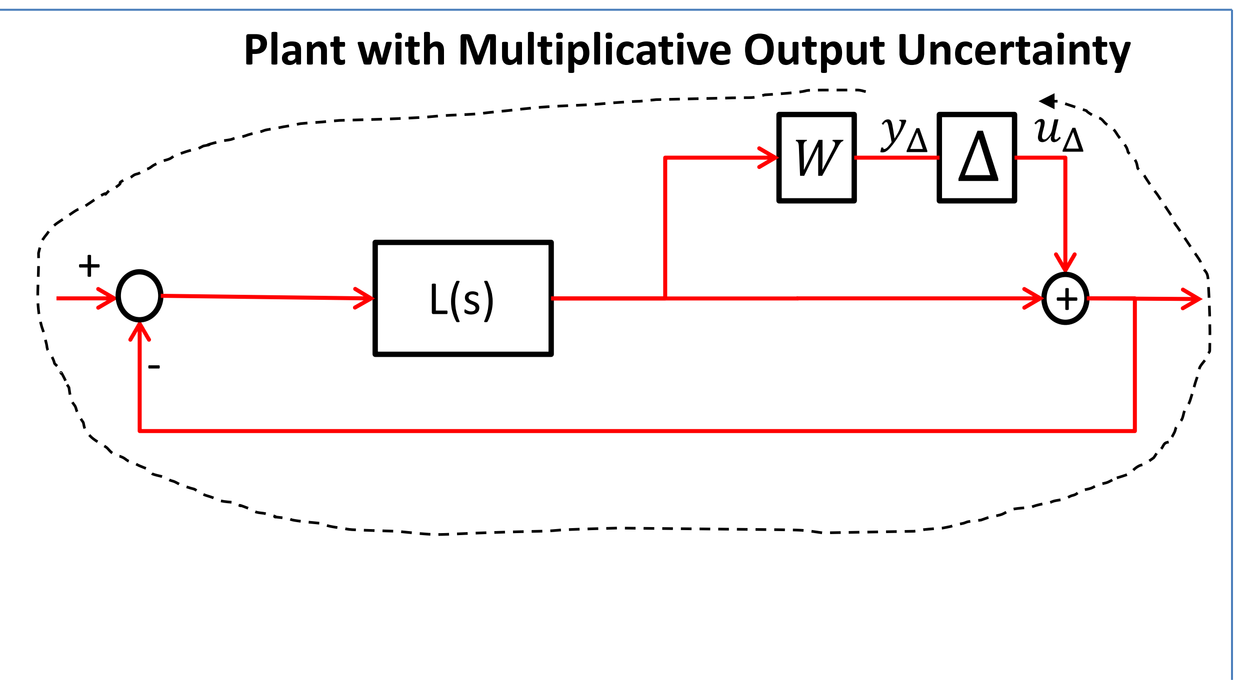
- The stability of the uncertain plant is analyzed by applying the Generalized Nyquist stability criterion to an alternative open loop: $M(s)\Delta$, not $L(s)$
  - Form $M(s)\Delta$ by breaking loop before and after uncertainty block, Δ
  - Assume closed loop system with $L(s)$ and negative feedback is stable

Plant with Multiplicative Output Uncertainty



How the $M\Delta$ loop is formed

- The transfer function is computed by going counterclockwise around the loop from all y_Δ's to all u_Δ's



Form M

$$\frac{y_\Delta}{u_\Delta} = M = -WL(I+L)^{-1}$$

Form $M\Delta$

$$M\Delta = -WL(I+L)^{-1}\Delta$$

Note: For output multiplicative uncertainty, the $M\Delta$ is an open loop system encasing the closed loop $L(s)$ system with negative feedback.
Generalized Nyquist stability criterion can be used on this loop

So how is μ conceptually computed?

- The μ is computed by finding the gain on the $M\Delta$ loop at all frequencies ω for which

$$\det(I + \epsilon(\omega) M(\hat{j}\omega)\Delta) = 0$$

Application of the generalized Nyquist stability criterion

- Where for

Unmodeled dynamics	Complex uncertainty	Parametric real uncertainty
$\|\Delta\|_\infty \leq 1$	$\lvert\Delta\rvert \leq 1$	$-1 < \Delta < 1$

- The μ is then defined as

$$\mu(\omega) := \frac{1}{\epsilon(\omega)} \; for \; all \; \omega$$

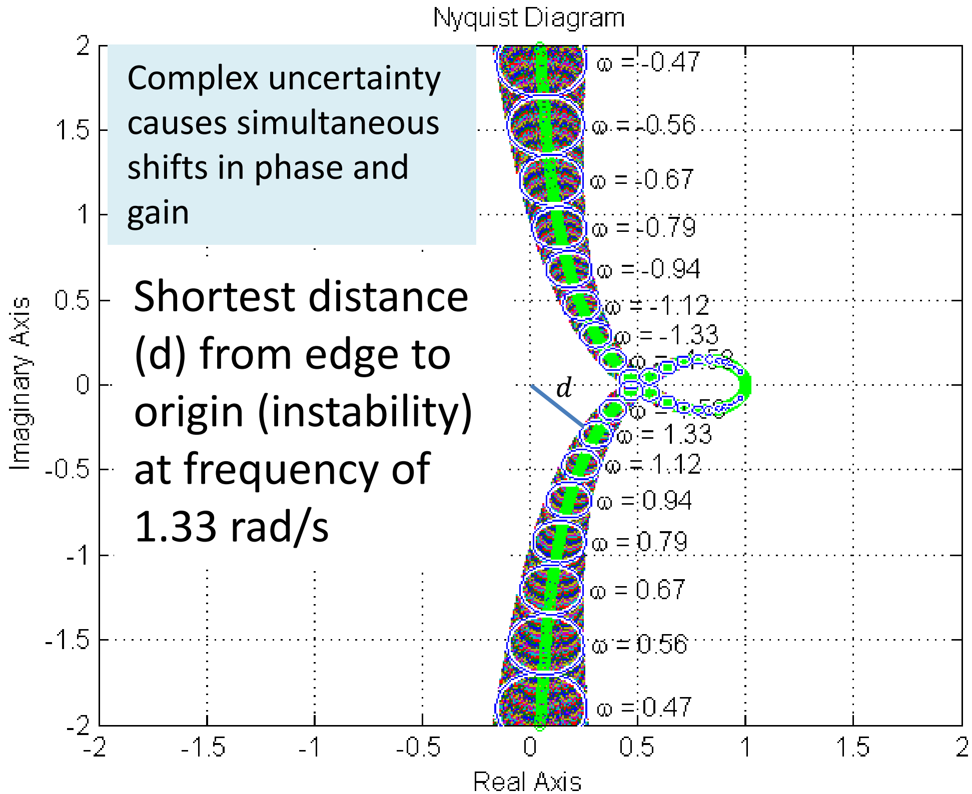
There are approximations of lower and upper bounds on ϵ (or μ) – NP hard

- The μ represents the maximum gain on all uncertainty loops over all all frequencies the system can tolerate before the system can no longer be guaranteed stable

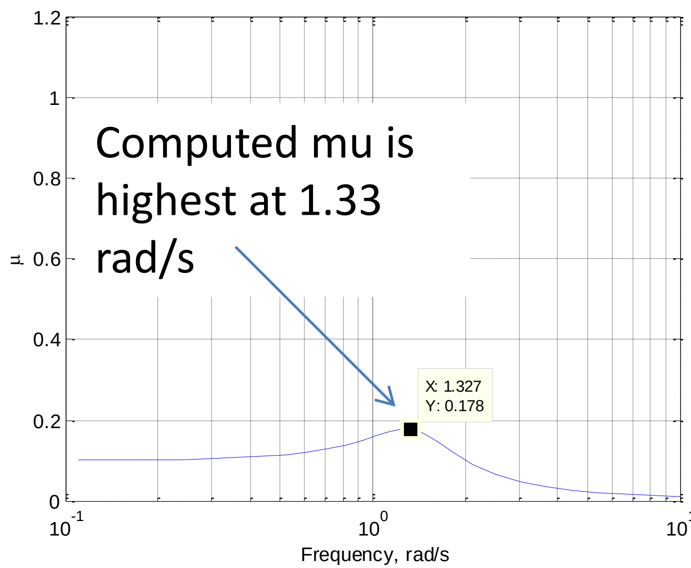
How μ compares to a Nyquist chart for uncertain plant

Nyquist plot of $\det(I + L_p(s))$
with 10% multiplicative complex uncertainty

$$L_p(s) = (I + 0.1 * \Delta)\frac{\frac{1}{s}(-s+2)}{s+2}$$



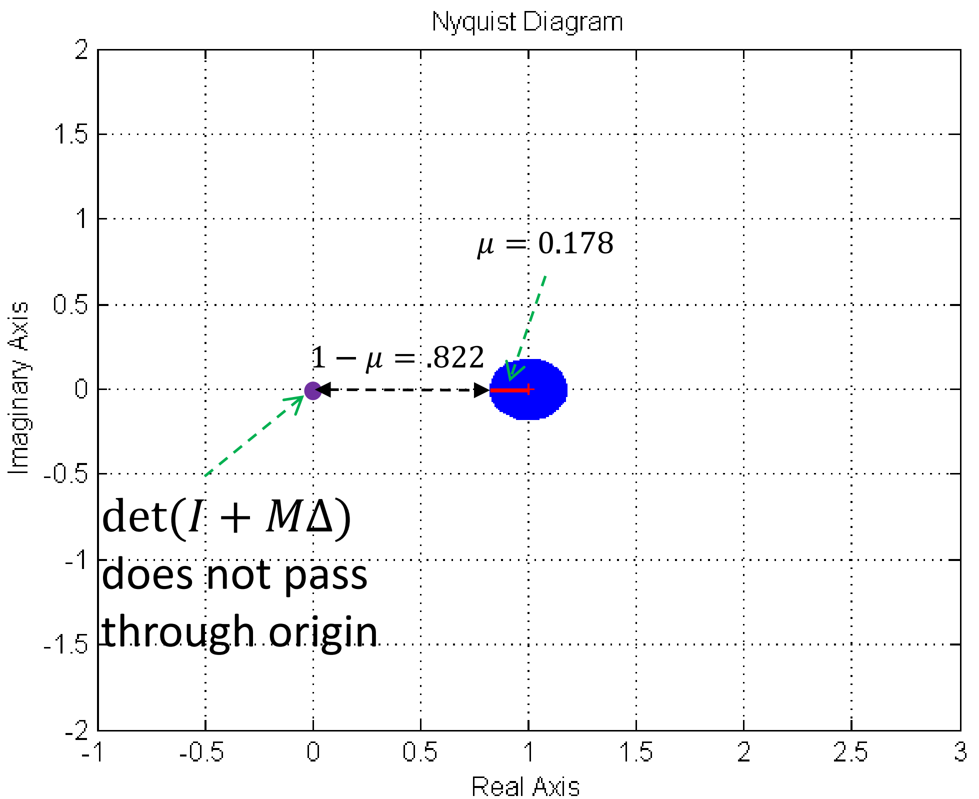
Mu analysis with
10% multiplicative complex uncertainty



μ is maximum at the frequency where the uncertainty bubble is closest to the origin

What does $\det(I + M\Delta)$ look like?

- The $det(I + M\Delta)$ is sampled 1,000 times and plotted in blue
  - The magnitude by which $M\Delta$ can be multiplied and the system remain stable (i.e. determinant not pass through origin) is $\epsilon = \frac{1}{\mu} = \frac{1}{0.178}$



So what does μ tell us then?

- The inverse of the maximum μ is the smallest amplitude by which all modeled uncertainties may be multiplied by before the system is no longer guaranteed stable
  - It is a margin, similar to gain and phase margin, but simultaneously accounting for both or unmodeled dynamics
  - The computed μ from last slide is 0.178, so

$$Tolerable\ uncertainty = \frac{1}{\mu} * \mathrm{W} = \frac{1}{.178} * 0.1 = 0.56 = 56\%$$

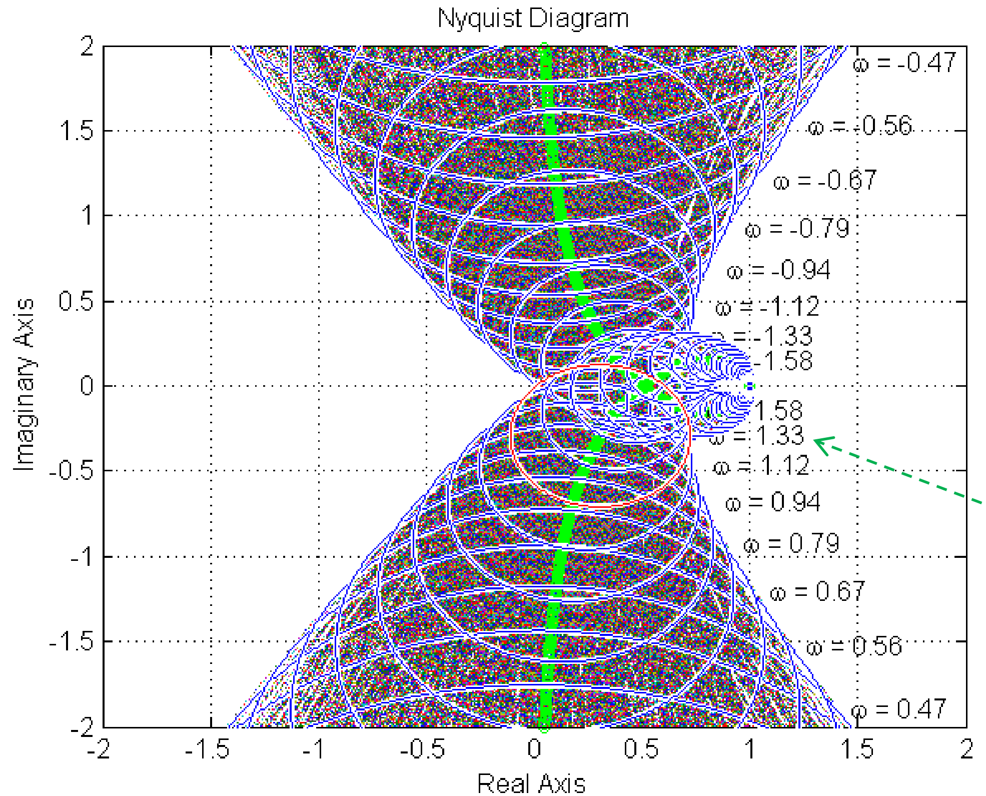
Modeled 10% uncertainty initially

  - 56% multiplicative complex uncertainty can be tolerated by the system at all frequencies

Using one μ computation to determine Margin

$$L_p(s) = (I + 0.56 * \Delta)\frac{\frac{1}{s}(-s+2)}{s+2}$$

Nyquist plot of det$(I + L_p(s))$
with 56% multiplicative complex uncertainty



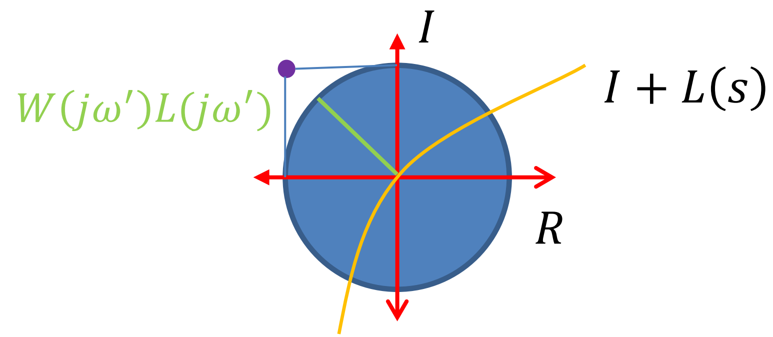
Blue circles are bounds of uncertainty at certain frequencies

Model is most sensitive at worst case frequency (red circle) of 1.33 rad/s as predicted and nearly intersects the origin

But a remaining question is what does 56% uncertainty physically mean?

How to use μ to Compute a Simultaneous Gain and Phase Margin

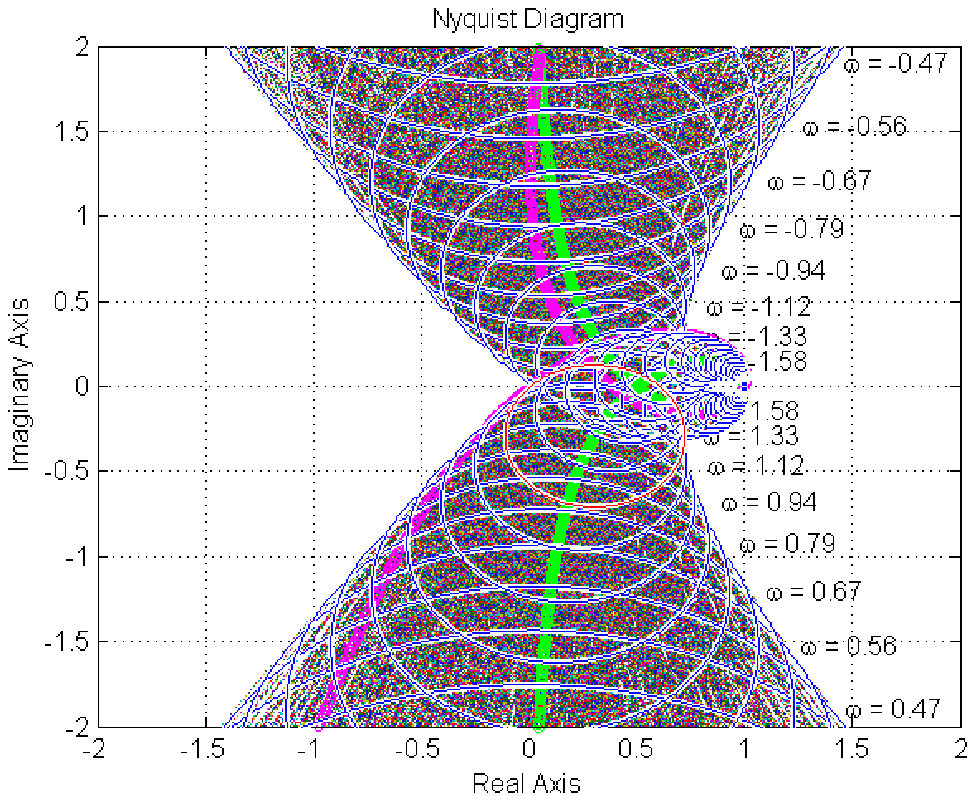
- Assumption
  - Analysis of a system with bounded multiplicative complex uncertainty
  - Assume: gain bound on complex uncertainty can be used to determine bound on gain margin
- $GM \leq 1 + \frac{1}{\mu(\omega')} * |W(\hat{j}\omega')|$



- Where
  - ω' is the worst case frequency
- Compute $L_p(s) = GM * L(s)$
- Perform margin analysis on $L_p(s)$ to find minimum simultaneous phase margin
  - For MIMO systems, diskmargin.m should be used but one can also apply gain to all loops and find a less conservative phase margin on the worst case loop

Simultaneous Gain and Phase Margin Computation from μ for SISO system

- Using this method I find a phase margin of 14.1 deg
- The new Nyquist plot is scaled and rotated
  - Magenta line: $1.56 * (1 + L(\hat{j}\omega)) * \exp(-14.1 * \frac{\pi}{180} * \hat{j})$



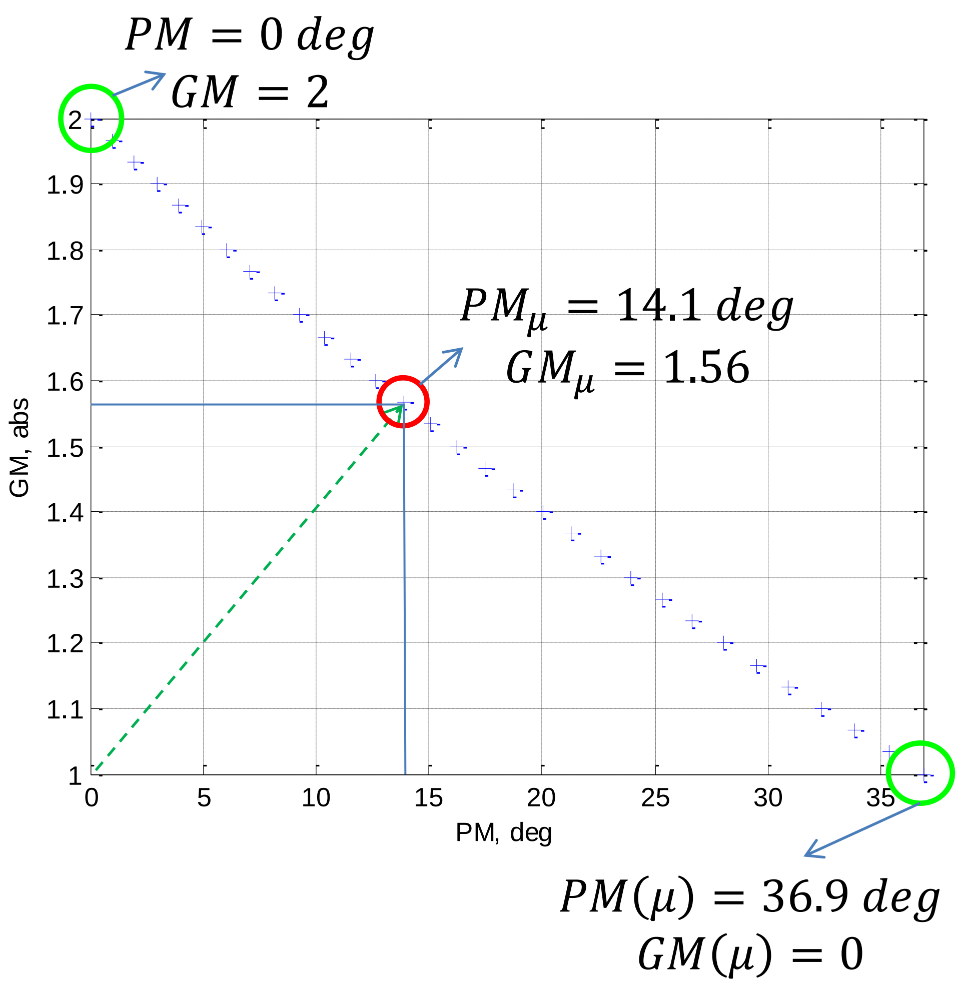
The line in magenta intersects the origin and is tangent to the red circle of uncertainty around the worst case frequency of 1.33 rad/s

Note: green line is the original Nyquist plot

Typical	➡	GM: 2 abs PM: 36.9 deg
Simultaneous	➡	GM: 1.56 abs PM: 14.1 deg

So what does the Gain and Phase Margin from μ mean?

- The phase and gain margins vary together in any system
- The phase and gain from μ represents the worst case amount of phase lag and gain variation which can be tolerated



The green circles represents what is given from a typical margin analysis. The μ can give a confidence point shown in red

How μ Serves to Augment Gain and Phase Margins

- The bode GM is 2 at 2 rad/s **OR** the bode PM is 36.9 deg at 1 rad/s
- The uncertainty margin is 56% which is equivalent to a GM of 1.56 **AND** a PM of 14.1 deg at the worst case frequency of 1.33 rad/s
- The uncertainty margin is conservative due to accounting for simultaneous gain and phase changes

The μ provides at least a third conservative point which could augment typical margin analyses methods.

Analyzing MIMO Systems

- Similar concepts can be applied to MIMO systems
  - Computing the simultaneous phase margin isn't as straightforward as one has to account for gains on all loops and phases on all other closed loops when analyzing each open loop
  - Much faster to use diskmargin.m
- The location of the uncertainty blocks have a strong impact on the robustness of the system
  - Additive uncertain plant is: $G_A = G + W_A\Delta$
  - Multiplicative Input is: $G_I = G(I + W_I\Delta)$
  - Multiplicative Output is: $G_O = (I + W_O\Delta)G$
- The robust stability margin for each type of uncertainty provides a relative assessment of aspects of the controller

MIMO Example of using μ to Compute Simultaneous Margins

- Boeing-767
  - LQG controller designed for longitudinal model

4.1 Boeing-767

The longitudinal and lateral B-767 state-space models are given below. The state vectors are:

$$\mathbf{x}_{\text{long}} = \begin{bmatrix} u \text{ (ft/s)} \\ \alpha \text{ (deg)} \\ q \text{ (deg/s)} \\ \theta \text{ (deg)} \end{bmatrix} \quad \mathbf{x}_{\text{lat}} = \begin{bmatrix} \beta \text{ (deg)} \\ p \text{ (deg/s)} \\ \phi \text{ (deg/s)} \\ r \text{ (deg)} \end{bmatrix} \tag{4.1}$$

$$\mathbf{u}_{\text{long}} = \begin{bmatrix} \delta_E \text{ (deg)} \\ \delta_T \text{ (\%)} \end{bmatrix} \quad \mathbf{u}_{\text{lat}} = \begin{bmatrix} \delta_A \text{ (deg)} \\ \delta_R \text{ (deg)} \end{bmatrix} \tag{4.2}$$

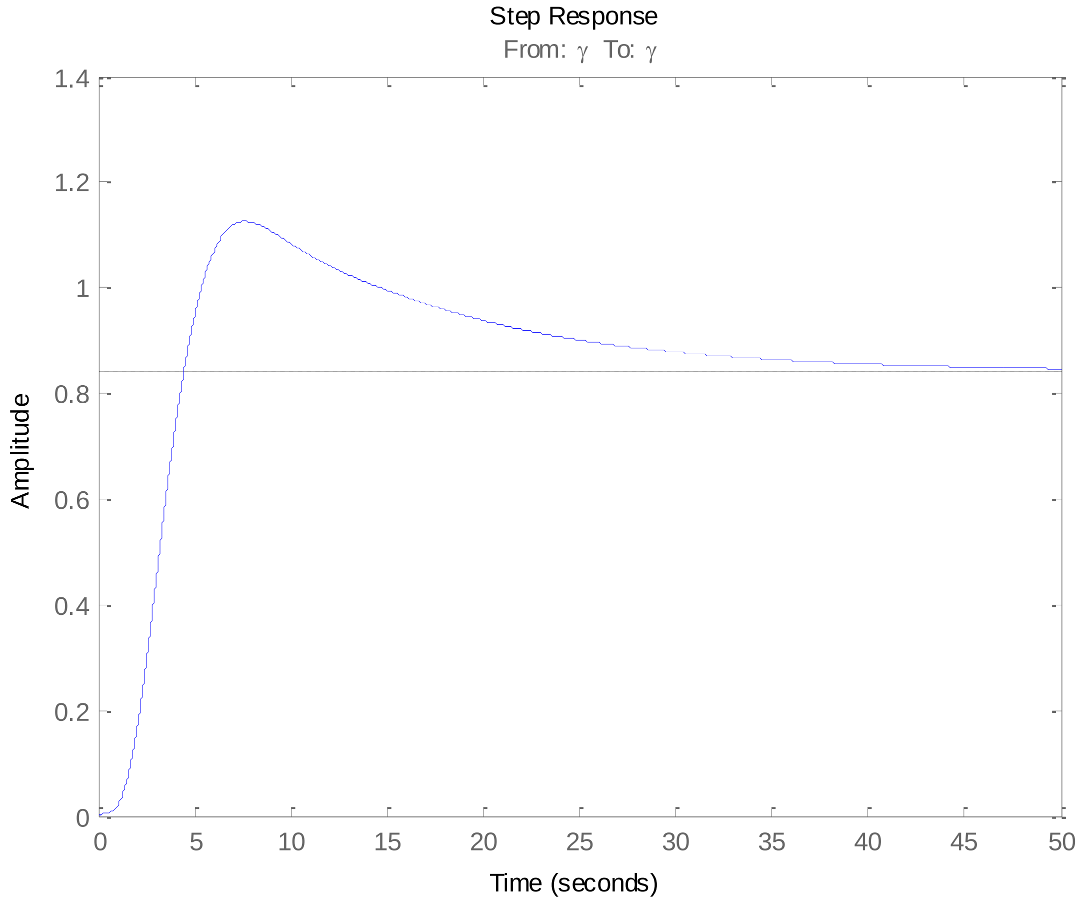
Equilibrium point:

Speed	$V_T = 890$ ft/s $= 980$ km/h
Altitude	$h = 35\,000$ ft
Mass	$m = 184\,000$ lbs
Mach-number	$M = 0.8$

Ref:
Mathematical models for control of aircraft and satellites, Thor I. Fossen, 2011, 2nd edition
http://www.airplanesgallery.com/boeing-767/

Longitudinal Characteristics
Tracking flight path angle

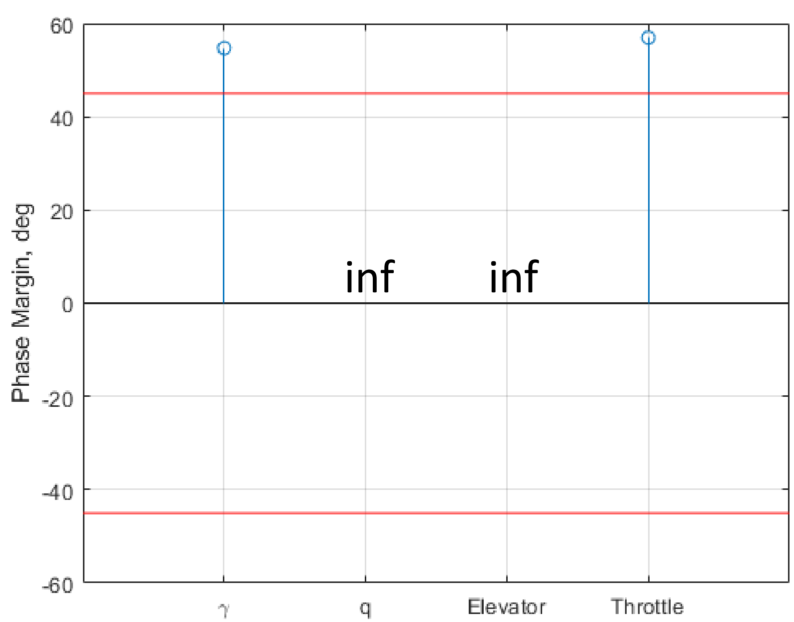
- Not a perfect design, but will serve purpose



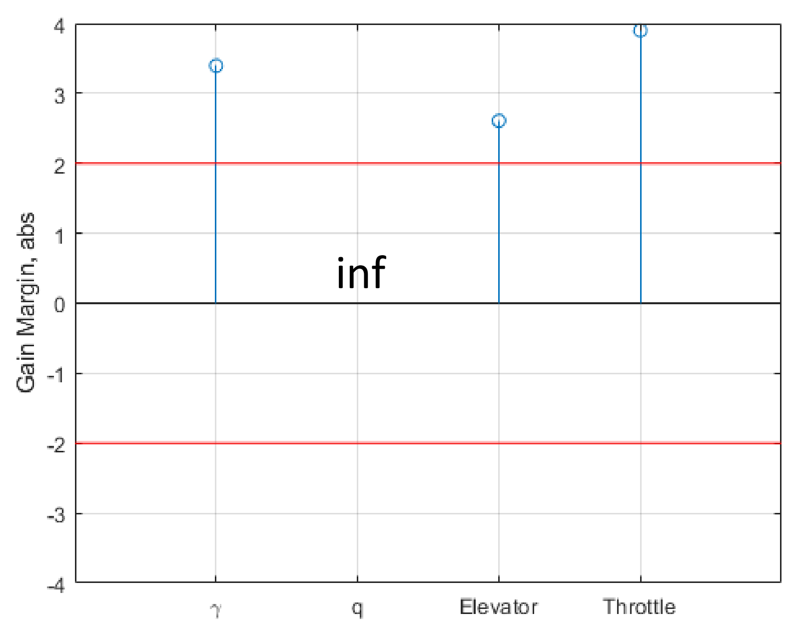
Longitudinal Margins Satisfied

- Margins of 45 deg. and 6 dB (or 2) were desired
  - Computed by closing all loops and opening one loop at a time
  - Lowest margin in any channel is GM of 2.6 and PM of 54.6 deg.

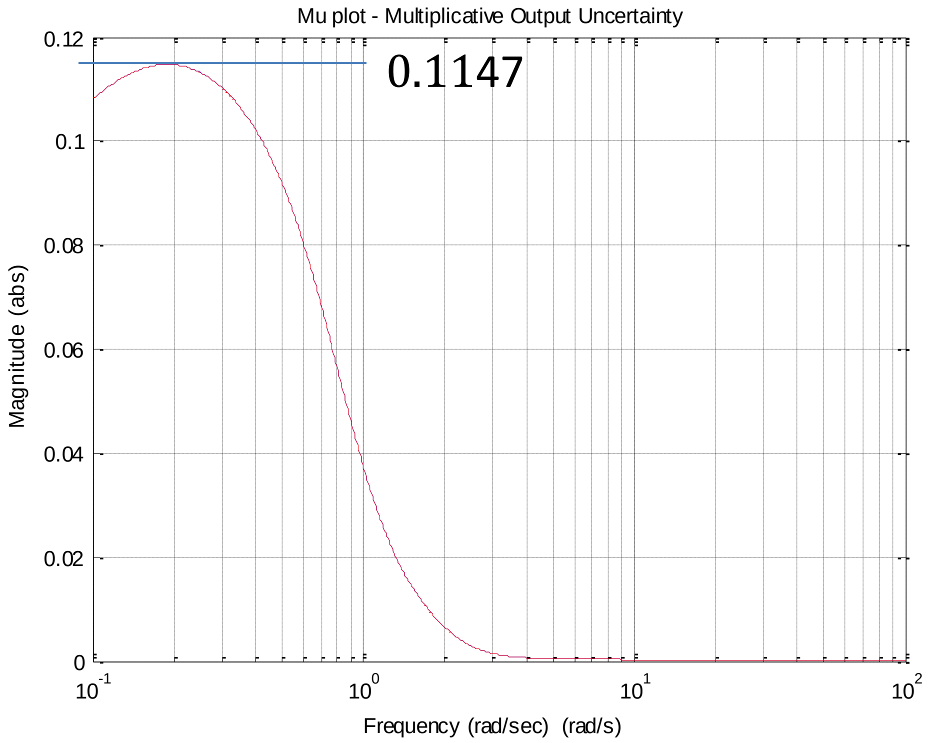
Phase Margin Analysis



Gain Margin Analysis



Simultaneous Margin Computation from Output Multiplicative Complex Uncertainty



Simultaneous GM (output)

$$GM\text{: } 1+\frac{|W|}{\mu(\omega')}=1+\frac{0.1}{0.1147}=1.87$$

Worst case PM (output)

Perform standard margin analysis for G(s)*I*1.87*K(s) and K(s)G(s)*I*1.87

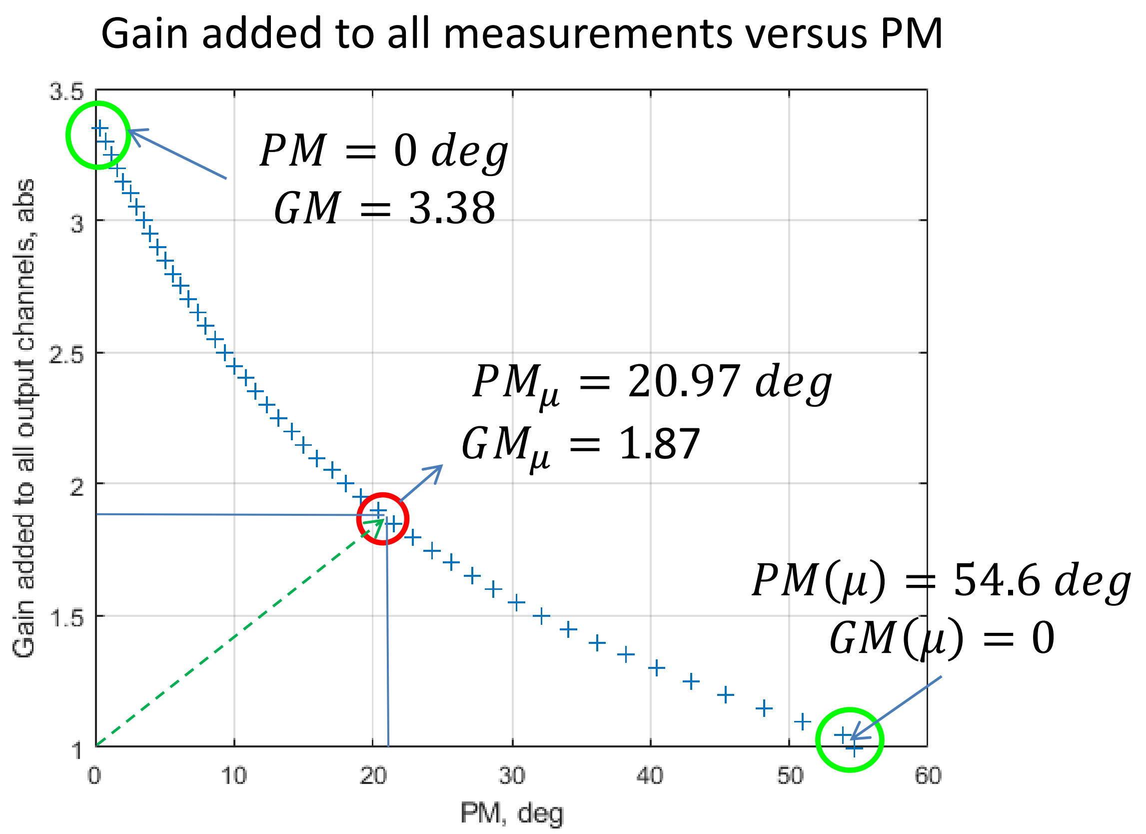
PM: 20.97 deg

Pitch rate	⇨	GM: inf abs PM: inf deg
Flight Path	⇨	GM: 3.38 abs PM: 54.6 deg
Simultaneous margin	⇨	GM: 1.87 abs PM: 20.97 deg

Simultaneous gain and phase margin from worst case loop

Effect of Varying Measurement Gains

- Gain is applied simultaneously to pitch rate and gamma
- Variation of gain and PM may not always be linear
  - Sensitivity to gain or phase
- The phase margin and gain variation computed from μ represent the worst case
  - As in the SISO case

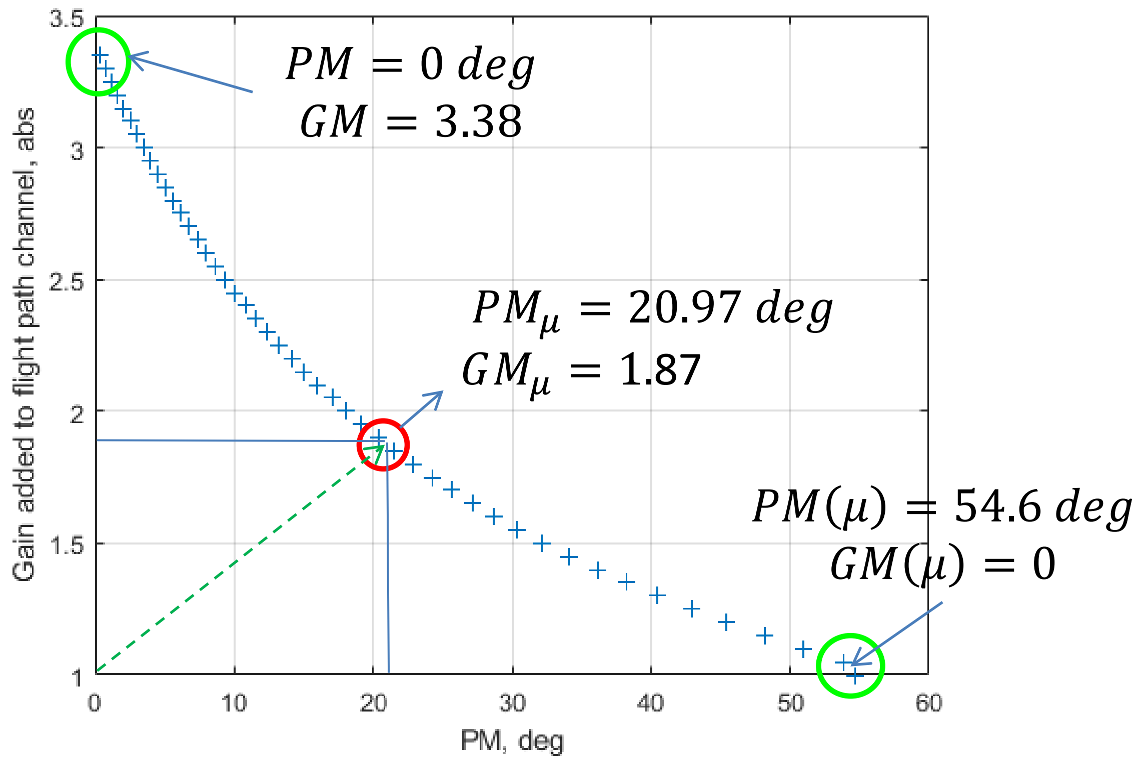


Observation: Simultaneous margins from μ provides not only a third conservative point but also a measure of sensitivity. Notice the curvature.

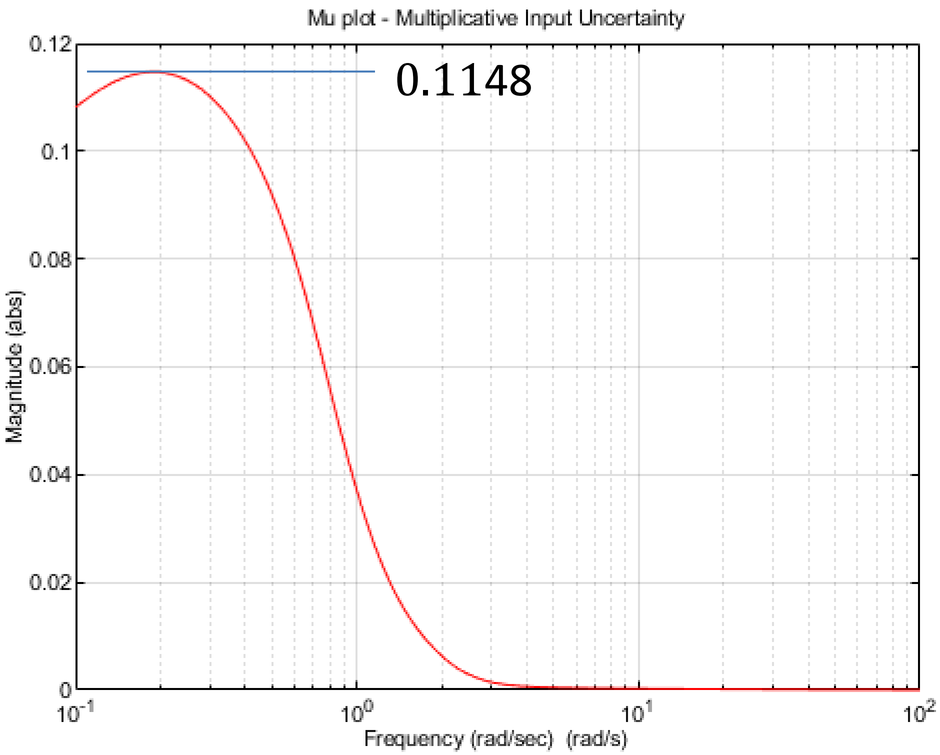
Effect of Varying only Flight Path Angle Gain

- Gain is applied only to flight path angle
- Considering similarity to previous chart, flight path angle is driving the μ computed from output uncertainty

Gain added to gamma channel versus PM



Simultaneous Margin Computation from Input Multiplicative Complex Uncertainty



Simultaneous GM (input)

$$GM\colon\ 1+\frac{|W|}{\mu(\omega')}=1+\frac{0.1}{0.1148}=1.87$$

Worst case PM (input)

Perform standard margin analysis for G(s)*I*1.87*K(s) and K(s)G(s)*I*1.87

PM: 21.03 deg

throttle margin ➡ GM: 3.9 abs, PM: 56.9 deg

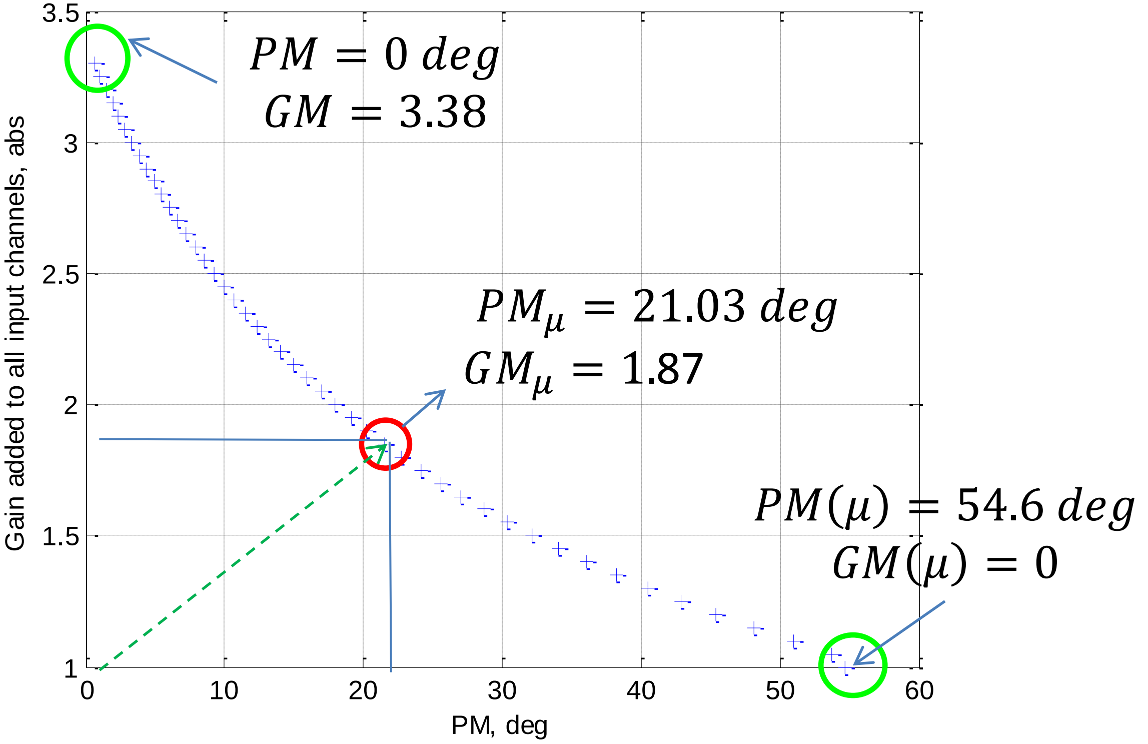
Elevator margin ➡ GM: 2.6 abs, PM: 54.6 deg

Simultaneous on all inputs ➡ GM: 1.87 abs, PM: 21.03 deg

Varying Control Effector Gains

- Gain is added to both elevator and throttle input channels
- GM and PM from μ is now located on the points
- Nearly duplicates the case for output gain changes
  - This might not be the case for more complicated systems

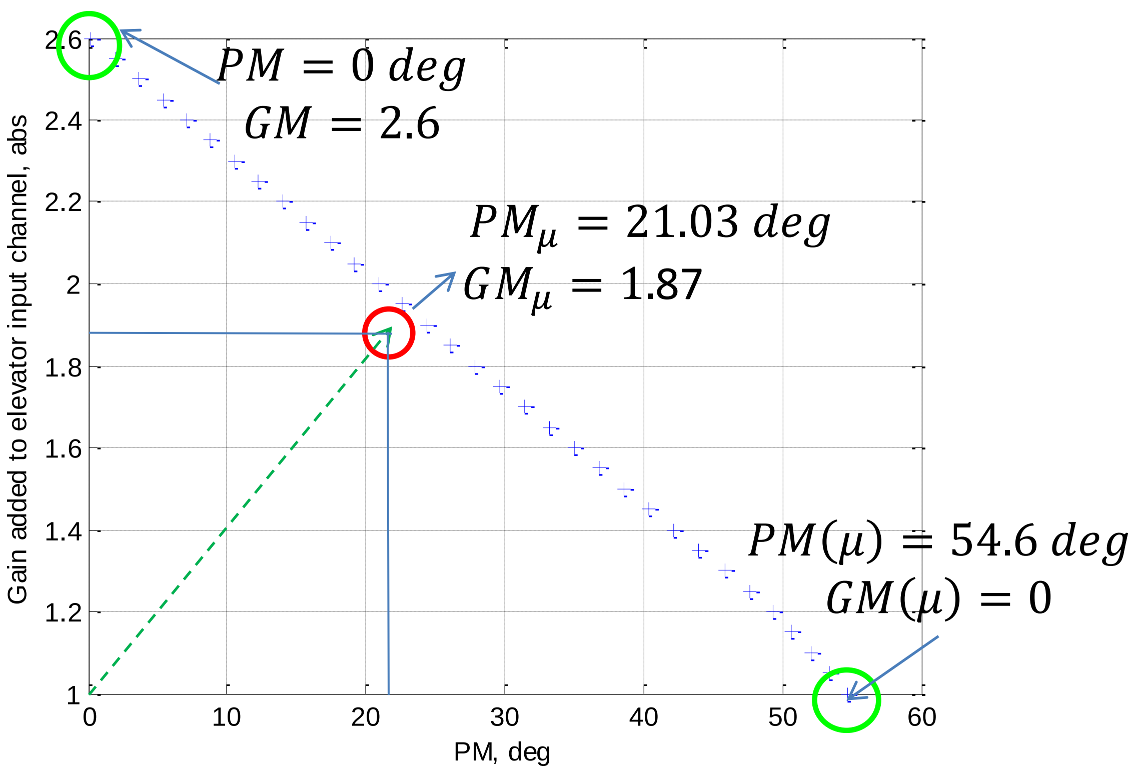
Gain added to all input channels



Varying only the Elevator Gain

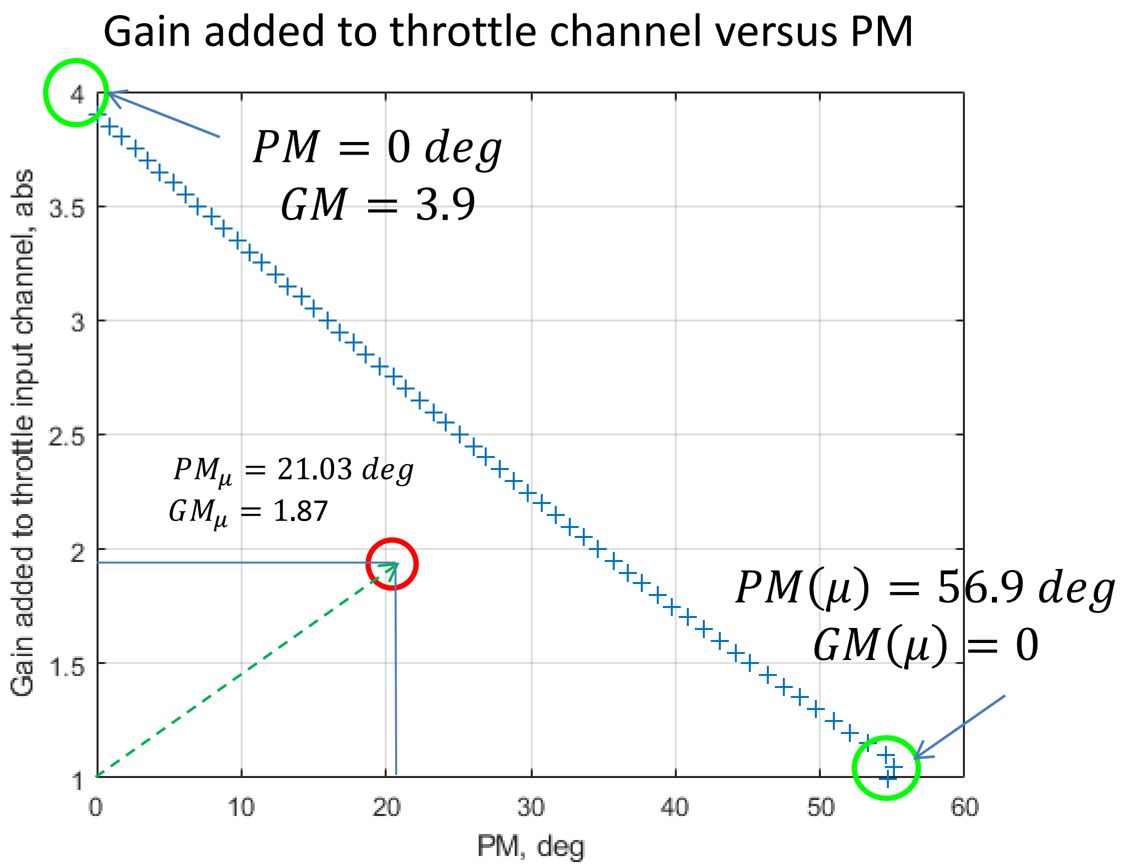
- Gain is added only to the input elevator channel
- GM and PM from μ is now located closer to the interior of the points
  - Suggests the elevator channel is likely driving μ computed from input uncertainty

Gain added to elevator channel versus PM



Varying only the throttle Gain

- Gain is added only to the input throttle channel
- GM and PM from μ is located much farther from the points
  - Suggests the throttle channel is less involved
  - The loop coupling of elevator and throttle may have created parabolic behavior



Conclusions

- μ or robust stability margins computed from μ may help quantify if
  - the system is sensitive to simultaneous gain and phase change
  - loop coupling is present due to frequency separation problems
  - The system is sensitive at a particular frequency known to be excited in the flight envelope
  - One uncertainty type is worse than others for MIMO systems
- μ is also an excellent tool to compare the robustness of one controller to another but it is hard to establish a standard with it
  - Typically the more loops the controller has the less simultaneous loop uncertainty the system can tolerate so robust stability margins must be relatively assessed against the same system
- Some flight programs may benefit from the additional knowledge gained by computing robust stability margins

Questions?